THE BAG MODEL OF NUCLEI

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Abstract

The basic assumptions and the general results of our bag model for nuclei are presented in detail. Nuclei are considered in a unified integration of the mean field theory and the MIT bag model.

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Chapter 1

The Mean Field Theory of Nuclei and The Bag Model of Baryon

1.1 Mean Field Theory of Nuclei (MFT)

Two model field theories of the nuclear system were studied in detail by Serot and Walecka [1]. The first is based on baryons and neutral scalar and vector mesons (model QHD-I). The quanta of the fields are the nucleons (n, p) and the sigma (σ) and omega (ω) mesons. The neutral scalar meson couples to the scalar density of baryons through $g_s\overline{\psi}\psi\phi$ and the neutral vector meson couples to the conserved baryon current through $g_v\overline{\psi}\gamma_\mu\psi V^\mu$.

The Lagrangian density for this model is

$$\begin{aligned} \mathcal{L} &= \overline{\psi}\left[\gamma^\mu\left(i\partial_\mu - g_v V_\mu\right) - \left(M - g_s\phi\right)\right]\psi \\ &\quad + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_v^2 V_\mu V^\mu, \end{aligned} \tag{1.1}$$

where

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu.$$

Lagrange's equations yield the field equations:

$$\left[\gamma^\mu\left(i\partial_\mu - g_v V_\mu\right) - \left(M - g_s\phi\right)\right]\psi = 0, \tag{1.2}$$

$$(\partial_\mu\partial^\mu - m_s^2)\phi = g_s\overline{\psi}\psi, \tag{1.3}$$

$$\partial_\mu F^{\mu\nu} - m_v^2 V^\nu = g_v\overline{\psi}\gamma^\nu\psi, \tag{1.4}$$

Eq. (1.2) is the Dirac equation with the scalar and vector fields. Eq. (1.3) is simply the Klein-Gordon equation with a scalar source. And Eq. (1.4) looks like massive QED with the conserved baryon current,

$$A^\nu = \overline{\psi}\gamma^\nu\psi. \tag{1.5}$$

The energy-momentum tensor is:

$$
\begin{aligned}
T^{\mu\nu} &= \frac{1}{2}\left[-\partial_\lambda\phi\partial^\lambda\phi + m_s^2\phi^2 + \frac{1}{2}F_{\lambda\sigma}F^{\lambda\sigma} - m_v^2 V_\lambda V^\lambda\right]g^{\mu\nu} \\
&\quad + i\overline{\psi}\gamma^\mu\partial^\nu\psi + \partial^\mu\phi\partial^\nu\phi + \partial^\nu V_\lambda F^{\lambda\mu}.
\end{aligned}
\tag{1.6}
$$

Lagrange's equations ensure that this tensor is conserved and satisfies $\partial_\mu T^{\mu\nu} = 0$. It follows that the energy-momentum P^ν defined by

$$
P^\nu = \int d^3x\, T^{0\nu}
\tag{1.7}
$$

is a constant of motion.

We observe that at high baryon density, the scalar and vector field operators can be replaced by their expectation values, which then serve as classical, condensed fields in which the baryons move,

$$
\phi \;\rightarrow\; \langle\phi\rangle \equiv \phi_0,
\tag{1.8}
$$
$$
V_\mu \;\rightarrow\; \langle V_\mu\rangle \equiv \delta_{\mu 0} V_0.
\tag{1.9}
$$

For a static, uniform system, the quantities ϕ_0 and V_0 are constants independent of x_μ. Rotational invariance implies that the expectation value $\langle\vec{V}\rangle$ vanishes.

The condensed, constant, classical fields ϕ_0 and V_0 are directly related to the baryon sources. The source for V_0 is simply the baryon density $\rho_A = A/V$. Since the baryon current is conserved, the baryon number

$$
A = \int_V d^3x\, \overline{\psi}\psi
\tag{1.10}
$$

is a constant of motion. For a uniform system of A baryons in a volume V, the baryon density is also a constant of motion. In contrast, the source for ϕ_0 involves the expectation value of the Lorentz scalar density $\langle\overline{\psi}\psi\rangle \equiv \phi_0$. This quantity is dynamical that must be calculated self-consistently using the thermodynamic argument that an isolated system at fixed A and V (and zero temperature) will minimize its energy,

$$
\frac{\partial E(A, V; \phi_0)}{\partial\phi_0} = 0.
\tag{1.11}
$$

In the MFT, the Lagrangian density is

$$
\begin{aligned}
\mathcal{L}_{\mathrm{MFT}} &= \overline{\psi}\left[i\gamma^\mu\partial_\mu - g_v\gamma^0 V_0 - (M - g_s\phi_0)\right]\psi \\
&\quad - \frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_v^2 V_0^2.
\end{aligned}
\tag{1.12}
$$

Hence, the Dirac equation is linear,

$$
\left[i\gamma^\mu\partial_\mu - g_v\gamma^0 V_0 - (M - g_s\phi_0)\right]\psi = 0,
\tag{1.13}
$$

and may be solved directly. We seek normal-mode solutions of the form $\psi(x^\mu) = \psi(\vec{x})e^{-iEt}$. This leads to

$$
\begin{aligned}
H\psi(\vec{x}) &= E\psi(\vec{x}), &(1.14)\\
H &\equiv [-i\vec{\alpha}.\nabla + g_v V_0 + \beta(M - g_s\phi_0).
\end{aligned}
$$

The effective mass M^* is defined by

$$M^* = M - g_s\phi_0. \tag{1.15}$$

The condensed scalar field ϕ_0 thus serves to shift the mass of the baryons. Evidently, the condensed vector field V_0 shifts the frequency (or energy) of the solutions,

$$E = g_v V_0 + E^*. \tag{1.16}$$

In the MFT, the energy-momentum tensor is

$$T^{\mu\nu}_{\mathrm{MFT}} = i\overline{\psi}\gamma^\mu\partial^\nu - \frac{1}{2}\left(m_v^2 V_0^2 - m_s^2\phi_0^2\right)g^{\mu\nu}. \tag{1.17}$$

It then follows that

$$P^\nu = \int_V d^3x\, T^{0\nu}. \tag{1.18}$$

The second nuclear model is more realistic. To discuss nuclei with $Z \neq N$, it is necessary to extend model QHD-I to include ρ mesons, which couple to the isovector current, and the coulomb interaction.

Since we are also interested in comparing quantitative predictions with experiment, we will extend the model to include the ρ meson and photon fields (QHD-II). The Lagrangian density is

$$
\begin{aligned}
\mathcal{L} = {} & \overline{\psi}\left[\gamma^\mu\left(i\partial_\mu - g_v V_\mu - \frac{1}{2}g_\rho\vec{\tau}.\vec{b}_\mu - \frac{1}{2}e(1+\tau_3)A_\mu\right) - (M - g_s\phi)\right]\psi\\
& + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_v^2 V_\mu V^\mu\\
& - \frac{1}{4}\vec{G}_{\mu\nu}.\vec{G}^{\mu\nu} + \frac{1}{2}m_\rho^2\vec{b}_\mu.\vec{b}^\mu - \frac{1}{4}H_{\mu\nu}H^{\mu\nu},
\end{aligned}
\tag{1.19}
$$

where

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu,\\
\vec{G}_{\mu\nu} &= \partial_\mu\vec{b}_\nu - \partial_\nu\vec{b}_\mu,\\
H_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu.
\end{aligned}
$$

The field equations (1.2 to (1.4) must also be extended to include contributions from the rho and photon fields,

$$
\begin{aligned}
(\partial_\mu\partial^\mu - m_\rho^2)\vec{b}_\mu &= g_\rho\overline{\psi}\vec{\tau}\gamma_\mu\psi, &(1.20)\\
\Box A_\mu &= e\overline{\psi}\frac{1+\tau_3}{2}\gamma_\mu\psi, &(1.21)\\
\left[\gamma^\mu\left(i\partial_\mu - g_v V_\mu - \frac{1}{2}g_\rho\vec{\tau}.\vec{b}_\mu - \frac{1}{2}e(1+\tau_3)A_\mu\right) - (M - g_s\phi)\right]\psi &= 0 &(1.22)
\end{aligned}
$$

If $N \neq Z$, the neutral field ρ^0 corresponding to $b_\mu^{(3)}$ can develop a classical, constant ground-state expectation value in nuclear matter according to

$$\langle b_\mu^{(j)} \rangle = \delta_{\mu 0} \delta^{j3} b_0. \tag{1.23}$$

The baryon field, however, obeys a Dirac equation analogous to (1.13) namely,

$$\left[i\gamma^\mu \partial_\mu - g_v \gamma^0 V_0 - \frac{1}{2} g_\rho \tau_3 \gamma^0 b_0 - \frac{1}{2} e(1 + \tau_3) \gamma^0 A_0 - (M - g_s \phi_0) \right] \psi = 0 \tag{1.24}$$

Although the baryon field is still an operator, the meson fields are classical; hence, Eq. (1.24) is linear, and we may seek normal-mode solutions of the form $\psi(x^\mu) = \psi(\vec{x}) e^{-iEt}$. This leads to

$$
\begin{aligned}
h\psi(\vec{x}) &= E\psi(\vec{x}), \\
h &\equiv \left[-i\vec{\alpha}.\nabla + g_v V_0 + \frac{1}{2} g_\rho \tau_3 b_0 + \frac{1}{2} e(1 + \tau_3) A_0 + \beta \left(M - g_s \phi_0 \right) \right],
\end{aligned} \tag{1.25}
$$

which defines the single-particle Dirac Hamiltonian h.

The single-particle wave functions in a central, parity-conserving field may be written as

$$\psi_\alpha(\vec{x}) = \psi_{nkm\tau}(\vec{x}) = \begin{pmatrix} \frac{G_{nk\tau}(r)}{r} \Phi_{km} \\ i \frac{F_{nk\tau}(r)}{r} \Phi_{-km} \end{pmatrix} \zeta_\tau. \tag{1.26}$$

The equations for the baryon wave functions follow immediately upon substituting (1.26) into (1.25):

$$\frac{dG_a}{dr} + \frac{k}{r} G_a - \left[E_a - g_v V_0 - \frac{1}{2} g_\rho \tau_a b_0 - \frac{1}{2} e(1 + \tau_a) A_0 + M - g_s \phi_0 \right] F_a = 0, \tag{1.27}$$

$$\frac{dF_a}{dr} - \frac{k}{r} G_a + \left[E_a - g_v V_0 - \frac{1}{2} g_\rho \tau_a b_0 - \frac{1}{2} e(1 + \tau_a) A_0 - M + g_s \phi_0 \right] G_a = 0. \tag{1.28}$$

For a given set of meson fields, the Dirac equations (1.27) and (1.28) may be solved by integrating outward from the origin and inward from large r, matching the solutions at some intermediate radius to determine the eigenvalues E_a. Analytic solutions in the regions of large and small r allow the proper boundary conditions to be imposed.

1.2 Relativistic Nuclear Physics

Relativistic nuclear physics was first pioneered and developed by Shakin and Celenza [2]. The systematic application of this theory is able to resolve some long-standing puzzles in the theory of nuclear structure such as the binding energy and the saturation density of nuclear matter, the effective force in nuclei, and the nucleon self-energy for bound and

continuums. Its success is based upon the use of the Dirac equation for the description of motion of a nucleon. The potentials appearing in the Dirac equation are assumed to contain large (Lorentz) scalar and vector fields.

The scalar fields enter the Dirac equation in the same way as the nucleon mass. Since these fields are quite large ($-400\,\text{MeV}$), they have the effect of including a major reduction of the nucleon mass when the nucleon is in the nuclear matter. It is the description of this change of mass of the nucleon that is an essential element in the success of the relativistic approach. It is further necessary to understand that the vector field seen by a nucleon is large and repulsive, so that the energy of the nucleon in the nuclear matter does not differ very much from the energy of a nucleon moving in the weak fields which appear in the standard Schrodinger description. More precisely, the description relation relating the energy and momentum of a nuclear quasiparticle is similar to that of the Schrodinger theory. Therefore, the system may be said to exhibit "hidden" relativity.

Indeed, for decades the Schrodinger approach to nuclear structure physics provided a reasonably satisfactory model of nuclear dynamics. It is only in the last decade that the true relativistic features of the system have become apparent.

We write the Dirac equation for a nucleon in the nuclear matter as

$$[\vec{\alpha}.\vec{p} + \gamma^0 m + V(\vec{p})]\phi(\vec{p},s) = \epsilon\phi(\vec{p},s),$$

where $V(\vec{p})$ is the potential. It will be useful to introduce the self-energy $\Sigma(\vec{p}) = \gamma^0 V(\vec{p})$ and rewrite this equation as

$$[\vec{\gamma}.\vec{p} + m + \Sigma(\vec{p})]\phi(\vec{p},s) = \gamma^0\epsilon\phi(\vec{p},s).$$

Now let us assume that the self-energy is of the form

$$\Sigma(\vec{p}) = A + \gamma^0 B,$$

so that we have

$$[\vec{\gamma}.\vec{p} + (m + A)]\phi(\vec{p},s) = \gamma^0(\epsilon - B)\phi(\vec{p},s).$$

A positive-energy spinor solution of this equation is

$$\phi(\vec{p},s) = \left(\frac{\widetilde{m}}{\widetilde{E}(\vec{p})}\right)^{1/2} u(\vec{p},s,\widetilde{m}),$$

where

$$u(\vec{p},s,\widetilde{m}) = \frac{\widetilde{E}(\vec{p}) + \widetilde{m}}{2\widetilde{m}} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}.\vec{p}}{\widetilde{E}(\vec{p})+\widetilde{m}}\chi_s \end{pmatrix}.$$

Here $u(\vec{p},s,\widetilde{m})$ is the positive-energy solution of the Dirac equation without interaction, except for the fact that the nucleon mass m has been replaced by $\widetilde{m} = m + A$ and $\widetilde{E}(\vec{p}) = \sqrt{\vec{p}^2 + \widetilde{m}^2}$. The normalization chosen here is

$$u^\dagger u = \widetilde{E}/\widetilde{m},$$

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so that

$$\phi^\dagger \phi = 1.$$

We further note that the energy eigenvalue is

$$
\begin{aligned}
\epsilon &= B + \sqrt{\vec{p}^2 + \widetilde{m}^2} \\
&= m + B + A + \frac{\vec{p}^2}{2\widetilde{m}} + \cdots.
\end{aligned}
$$

Now, as we have mentioned, A is large and negative (-400 MeV) and B is large and positive (300 MeV). Therefore A and B largely cancel and the dispersion relation is essentially the same as that which one could find in a nonrelativistic model. The development of this simple relativistic nuclear model leads to two categories. The first we will call Dirac Phenomenology. This category is distinguished by having several free parameters which are adjusted to fit nuclear date. The second category will be called Relativistic Brueckner-Hartree-Fock (RBHF) theory and is characterized as having no free parameters other than those introduced in fitting free - space nucleon - nucleon scattering data. Interest in the development of the RBHF approximation grew out of the successful application of Dirac phenomenology to the description of nucleon - nucleus scattering data.

We have so far presented only the main idea and the materials necessary to our consideration. For those who are interested to the detailed results of the relativistic nuclear physics, please, see the monograph of Celenza and Shakin quoted above and the references herein.

1.3 Bag Model of Baryons

It is well accepted QCD is the theory of strong interactions. However, in general, QCD is never solvable: at low energies and small momenta transfer the running coupling constant $\alpha_s > 1$. The bag model, outlined for the first time by the group of M.I.T. theorists [3], is a phenomenological approach, in which two basic features of QCD are incorporated: asymptotic freedom and confinement.

The main assumption of the M.I.T. bag model states that, baryon is considered to be a bag of spherical shape, in which the constituent quarks move freely and are described by the Dirac equation

$$H\psi = i\frac{\partial \psi}{\partial t},$$

with the Hamiltonian

$$H = \vec{\alpha}.\vec{p} + \beta M.$$

Consider the case $k = -1$, which is the $S_{1/2}$ level. The solution of this equation has

the form

$$\psi_{n,-1}(\vec{r},t) = N_{n,-1} \begin{pmatrix} \sqrt{\frac{E+M}{E}} \; j_0\left(\frac{\omega r}{R}\right) \chi_{-1}^m \\ -i\sqrt{\frac{E-M}{E}} \; j_1\left(\frac{\omega r}{R}\right) \chi_1^m \end{pmatrix} e^{-iEt}.$$

If we parametrize the energy levels as

$$\widetilde{E}_{nk} = \omega_{nk}/R, \qquad \widetilde{E}_{nk} = \sqrt{E^2 - M^2},$$

the density of quarks is readily calculated as

$$J^0 = \overline{\psi}\gamma^0\psi \sim \left[j_0^2\left(\frac{\omega r}{R}\right) + \frac{E-M}{E+M} \, j_1^2\left(\frac{\omega r}{R}\right) \right] \theta_V,$$

where

$$\theta_V = \begin{cases} 1 & r \leq R \\ 0 & r > R. \end{cases}$$

Thus, the density certainly does not vanish at $r = R$. Clearly, although the lower component is suppressed for small r, it does make a sizeable contribution near the surface of the bag. Of course it is natural to ask whether this is not unusual in comparison with nonrelativistic experience, where $\psi(R)$ would be zero. However, such a solution would not be consistent with the linear Dirac equation. What counts is that there should be no current flow through the surface of the confining region. For example, in the MIT bag model it is required that

$$n_\mu \overline{\psi}\gamma^\mu\psi = 0$$

at the surface - where n_μ is a unit four vector normal to the surface of the confining region.

In the MIT bag model this condition is imposed through a linear boundary condition

$$i\gamma.n\psi = \psi$$

at the surface. This implies

$$\psi^\dagger = -i\psi^\dagger\gamma^\dagger.n,$$

and hence

$$\overline{\psi} = -i\overline{\psi}\gamma.n,$$

because

$$\gamma^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0.$$

Consider now the normal flow of current through the bag surface:

$$\begin{aligned} in_\mu J^\mu &= in_\mu \overline{\psi}\gamma^\mu\psi \\ &= (i\overline{\psi}\gamma.n)\psi = \overline{\psi}(i\gamma.n\psi) \\ &= -\overline{\psi}\psi = \overline{\psi}\psi = 0. \end{aligned}$$

Thus, it is not the density, but $\overline{\psi}\psi$ which should vanish at the boundary in the relativistic theory,

$$\overline{\psi}\psi\big|_{r=R} = \frac{E+M}{E}\, j_0^2(\omega) - \frac{E-M}{E}\, j_1^2(\omega) = 0.$$

That is, the matching condition is exactly equivalent to the linear boundary condition (l.b.c.) for the static spherical MIT bag,

$$i\gamma.n\psi = -i\gamma.\hat{r}\psi = \psi,$$

where

$$n^\mu = (0,\hat{r}).$$

We consider the energy-momentum tensor for a model,

$$T_V^{\mu\nu} = T^{\mu\nu}\theta_V,$$

and $T^{\mu\nu}$ is the familiar energy-momentum tensor for a free Dirac field

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi.$$

The condition for overall energy and momentum conservation is that the divergence of the energy-momentum tensor should vanish, and this is certainly true for $T^{\mu\nu}$, as is easily proven from the free Dirac equation

$$\partial_\mu T^{\mu\nu} = 0.$$

However, the fact that these quarks move freely only inside the restricted region of space V leads to problems. Indeed,

$$\partial_\mu\theta_V = n_\mu\Delta_s,$$

where Δ_s is a surface delta function

$$\Delta_s = -n.\partial(\theta_V).$$

In the static spherical case we find that Δ_s is simply $\delta(r-R)$. Putting all these together we obtain

$$\partial_\mu T_V^{\mu\nu} = i\overline{\psi}\gamma.n\partial^\nu\psi\Delta_s,$$

and using the l.b.c.

$$\partial_\mu T_V^{\mu\nu} = -\frac{1}{2}\, \partial^\nu(\overline{\psi}\psi)\big|_s \Delta_s = -Pn^\nu\Delta_s,$$

where P is the pressure exerted on the bag wall by the contained Dirac gas

$$P = -\frac{1}{2}\, n.\partial^\nu(\overline{\psi}\psi)\big|_s.$$

Clearly, this model violates energy-momentum conservation. Furthermore, this violation is an essential result of the confinement process.

The resolution of this problem, we add an energy density term $B\theta_V$ to the Lagrangian density. Then (since $T^{\mu\nu}$ involves $\mathcal{L}g^{\mu\nu}$) the new energy-momentum tensor $T^{\mu\nu}_{\text{MIT}}$ has the form

$$T^{\mu\nu}_{\text{MIT}} = (T^{\mu\nu} + Bg^{\mu\nu})\theta_V.$$

Therefore, the divergence of the energy-momentum tensor is

$$\partial_\mu T^{\mu\nu}_{\text{MIT}} = (-P + B)n^\nu \Delta_s,$$

which will vanish if

$$B = P = --\frac{1}{2}\, n.\partial^\nu (\overline{\psi}\psi)\big|_s .$$

Therefrom a relativistic bag model of nuclei will be proposed, the A-dependence of nuclear radius will be calculated and the Weizssacker formula for nuclear binding energy will be derived exactly if the corresponding parameters of the model are adequately fitted.

Chapter 2

Bag Model of Nuclei $(Z = N)$

2.1 Main Assumption

We consider the simplest possible case of A baryons moving inside a spherical volume of radius R, outside of which there is a pressure exerted on the nuclear surface.

Let us therefore begin with the Dirac equation for a baryon of mass M:

$$(i\gamma^\mu \partial_\mu - M - \Sigma)\psi(x_\mu) = 0, \tag{2.1}$$

where Σ is the baryon self-energy having the form

$$\Sigma = \phi + \gamma^\mu V_\mu. \tag{2.2}$$

Inserting (2.2) into (2.1) we obtain the equation

$$[\gamma^\mu(i\partial_\mu - V_\mu) - (M + \phi)]\psi(x_\mu) = 0. \tag{2.3}$$

Eq. (2.3) is nonlinear quantum field equation and its exact solution is very complicated. We have therefore made little progress by writing down this equation with a suitable method for solving it.

In the MFT,

$$\phi \quad \rightarrow \quad \langle\phi\rangle \equiv \phi_0, \tag{2.4}$$

$$V_\mu \quad \rightarrow \quad \langle V_\mu\rangle \equiv \delta_{\mu 0} V_0. \tag{2.5}$$

For a static, uniform system the quantities ϕ_0 and V_0 are constants. Hence, the Dirac equation is linear,

$$[i\gamma^\mu \partial_\mu - \gamma^0 V_0 - (M + \phi_0)]\psi(x_\mu) = 0, \tag{2.6}$$

and may be solved directly.

Our basic assumption is formulated: the nucleus A is considered to be a MIT bag, inside of which the motion of nucleon is described by the Dirac equation (2.6). The quantities ϕ_0 and V_0 can be determined only after we have fitted to experimental data.

Let us next consider the energy-momentum conservation for nuclear bag. For stable nuclei, there should be no current flow through the surface of the confining region. In the MIT bag model it is required that

$$in_\mu J^\mu = -\overline{\psi}\psi = \overline{\psi}\psi = 0 \tag{2.7}$$

at the surface, where n_μ is a unit four vector normal to the surface of the confining region. Thus, $\overline{\psi}\psi$ which should vanish at the boundary in a relativistic theory. The matching condition of the present model is exactly equivalent to the linear boundary condition for the static spherical MIT bag.

The Lagrangian density for the present model is

$$\mathcal{L} = \overline{\psi}[i\gamma^\mu\partial_\mu - \gamma^0 V_0 - (M + \phi_0)]\psi\theta_V + B\theta_S, \tag{2.8}$$

where $B\theta_S$ is a energy density term.

Then the energy-momentum tensor has the form

$$T_{\text{Bag}}^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi\theta_V + Bg^{\mu\nu}\theta_S, \tag{2.9}$$

where θ_V and θ_S define the bag volume and the surface

$$\theta_V = \begin{cases} 1 & r \leq R \\ 0 & r > R \end{cases}, \qquad \theta_V = \begin{cases} 0 & r < R \\ 1 & r = R \\ 0 & r > R \end{cases} \tag{2.10}$$

and B is the constant surface tension.

Therefore, the divergence of the energy-momentum tensor is

$$\begin{aligned} \partial_\mu T_{\text{Bag}}^{\mu\nu} &= \left[\frac{1}{2}\, n.\partial(\overline{\psi}\psi)\big|_S + 2B\right]n^\nu\Delta_S \\ &= (-P_S + 2B)n^\nu\Delta_S. \end{aligned} \tag{2.11}$$

The condition for energy and momentum conservation is

$$\partial_\mu T_{\text{Bag}}^{\mu\nu} = 0, \tag{2.12}$$

hence

$$B = -\frac{1}{4}\, n.\partial(\overline{\psi}\psi)\big|_S, \tag{2.13}$$

where P_S is the pressure exerted on the bag wall by the contained A baryons.

E. (2.13) involves the square of the baryon fields, and is referred to as the nonlinear boundary condition of the MIT bag model of nuclei. Because of this condition the introduction of a constant surface tension B involves no new parameters.

Eq. (2.6) and the condition (2.13) constitute the basic ingredients of our model.

2.2 Wave Function for Baryon

The Dirac equation for the present model is linear and may be solved directly. We seek normal-mode solutions of the form

$$\psi(x_\mu) = \psi(\vec{r})e^{-iEt}\theta_V. \tag{2.14}$$

The Dirac equation then becomes

$$
\begin{aligned}
H\psi(\vec{r}) &= E\psi(\vec{r}), \\
H &\equiv [-i\vec{\alpha}.\nabla + V_0 + \beta(M + \phi_0)].
\end{aligned}
\tag{2.15}
$$

The effective mass M^* is defined by

$$M^* = M + \phi_0. \tag{2.16}$$

The scalar field ϕ_0 thus serves to shift the mass of the baryons. Evidently, the vector field V_0 shifts the frequency (or energy) of the baryon,

$$E^* = E - V_0. \tag{2.17}$$

Hence, Eq. (2.15) becomes

$$(-i\vec{\alpha}.\nabla + \beta M^*)\psi(\vec{r}) = E^*\psi(\vec{r}). \tag{2.18}$$

The single-particle wave functions in a central, parity-conserving field may be written as

$$\psi_\alpha(\vec{r}) = \psi_{nkm\tau}(\vec{r}) = \left(\begin{array}{c} \frac{G_{nk\tau}(r)}{r}\Phi_{km} \\ i\frac{F_{nk\tau}(r)}{r}\Phi_{-km} \end{array} \right)\zeta_\tau. \tag{2.19}$$

Their angular momentum and spin parts are simply spin spherical harmonics

$$\Phi_{km} = \sum_{m_l,m_s} \langle lm_l\frac{1}{2}m_s|l\frac{1}{2}jm\rangle Y_{lm_l}\chi_{m_s}, \tag{2.20}$$

$$k = \left\{ \begin{array}{l} l = +(j+1) > 0 \\ -(l+1) = -(j+1) < 0 \end{array} \right., \tag{2.21}$$

where Y_{lm_l} is a spherical harmonic and χ_{m_s} is a two-component Pauli spinor. The label α, $\{\alpha\} = \{a; m\} = \{nk\tau; m\}$, specifies the full set of quantum numbers describing the single-particle solutions. Since the system is assumed spherically symmetric and parity conserving, α contains the usual angular-momentum and parity quantum numbers. ζ_τ is a two-component isospinor. The principal quantum number is denoted by n. The phase choice in (2.19) leads to real bound-state wave functions G and F for real potentials in Hamiltonian (2.15).

The equations for the baryon wave functions follow immediately upon substituting (2.19) into (2.18)[1]:

$$\left(\frac{d}{dr} + \frac{k}{r}\right) G - (E^* + M^*)F = 0, \tag{2.22}$$

$$\left(\frac{d}{dr} - \frac{k}{r}\right) F + (E^* - M^*)G = 0. \tag{2.23}$$

These equations contain all information about the static ground-state nucleus. They are coupled linear differential equations that may be solved exactly for a given set of potentials.

Consider the case $k = -1$ which is the level $S_{1/2}$. Eq. (2.22) implies

$$F = (E^* + M^*)^{-1} \left(\frac{d}{dr} - \frac{1}{r}\right) G, \tag{2.24}$$

so that defining

$$W^2 = E^{*2} - M^{*2}, \tag{2.25}$$

the equation for the upper component of $\psi_\alpha(\vec{r})$ is

$$\left(\frac{d^2}{dr^2} + W^2\right) G = 0 \tag{2.26}$$

The solution of this equation has the form

$$G(r) = C \sin Wr, \tag{2.27}$$

and hence [from Eq. (2.24)]

$$F(r) = C(E^* + M^*)^{-1}(W \cos Wr - \sin Wr/r). \tag{2.28}$$

The solutions of the Dirac equation (2.18) come from (2.19) is written as

$$\psi_\alpha(\vec{r}) = C \begin{pmatrix} j_0(Wr)\,\Phi_{-1m} \\ -i\frac{W}{E^*+M^*}\, j_1(Wr)\,\Phi_{1m} \end{pmatrix} \zeta_\tau. \tag{2.29}$$

The normalization condition that yields the numbers of baryons contained in the nucleus A,

$$\int d^3x \; \psi^\dagger \psi = A. \tag{2.30}$$

Now let us assume that the bag has a spherical shape with radius R. $\overline{\psi}\psi$ which should vanish at the boundary in a relativistic theory. Eq. (2.29) implies that [see Eq. (2.7)],

$$\overline{\psi}\psi\big|_{r=R} = j_0^2(WR) - \frac{W^2}{(E^* + M^*)^2} j_1^2(WR) = 0 \tag{2.31}$$

---

[1]We use $\vec{\sigma}.\nabla(G\Phi_{km}/r) = -(1/r)(d/dr + k/r)G\Phi_{-km}$, and a similar relation for F.

and hence

$$j_0(WR) = \sqrt{\frac{E^* - M^*}{E^* + M^*}}\, j_1^2(WR). \tag{2.32}$$

This is appropriate boundary condition for confined baryons. Thus, the boundary condition of the MIT bag model is used, which provided the eigenfrequency of baryon ω_a, if we parametrize the energy levels (wavenumber) as

$$W_a = \omega_a/R; \qquad \{a\} = \{nk\tau\}, \qquad k = -1, \tag{2.33}$$

where n is the principal quantum number and ω_a satisfies the equation [from Eq. (2.32)]

$$\tan \omega_a = \frac{\omega_a}{1 - M^*R - \sqrt{\omega_a^2 + M^{*2}R^2}}. \tag{2.34}$$

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Hence, the eigenvalues E_a may be determined by matching the solutions at some intermediate radius. Analytic solutions in the restricted region of space V allow the proper boundary conditions to be imposed. Taking into consideration (2.33) we get the energy spectra for baryon,

$$\begin{aligned} E_a &= \pm\sqrt{W_a^2 + M^{*2}} + V_0 \\ &= \pm\sqrt{\frac{\omega_a^2}{R^2} + M^{*2}} + V_0. \end{aligned} \tag{2.35}$$

For convenience, the sign $(-)$ drops out in what follows. The single-particle wave functions now has the form

$$\psi_\alpha(\vec{r}) = C \left(\begin{array}{c} j_0\left(\frac{\omega_a}{R}r\right)\Phi_{-1m} \\ -i\frac{W_a}{E_a^* + M^*}\, j_1\left(\frac{\omega_a}{R}r\right)\Phi_{1m} \end{array} \right) \zeta_\tau. \tag{2.36}$$

Given the general form of the solutions in (2.36), we may now evaluate the local baryon density. Assume that the nuclear ground state consists of filled shells up to some value of n and k. This is consistent with spherical symmetry and is appropriate for magic nuclei.

With these assumptions, the local density of baryons is readily calculated as

$$\begin{aligned} \rho_A &= \psi^\dagger \psi \theta_V \\ &= C^2 \left[j_0^2\left(\frac{\omega_a r}{R}\right) + \frac{E_a^* - M^*}{E_a^* + M^*}\, j_1^2\left(\frac{\omega_a r}{R}\right) \right] \theta_V. \end{aligned} \tag{2.37}$$

Substituting Eq. (2.37) into Eq. (2.30), we can calculate the normalization constant which is defined by Eq. (2.30) for $k = -1$,

$$C^2 = \frac{A}{4\pi R^3 j_0^2(\omega_a)} \left(\frac{E_a^* + M^*}{E_a^*} \right) \frac{E_a^*(E_a^* - M^*)R}{2E_a^{*2}R - 2E_a^* + M^*}. \tag{2.38}$$

_____

[2]We use (2.25) and (2.33).

Finally, the single-particle wave functions may be written as

$$\psi_\alpha(\vec{r}) = N \left(\begin{array}{c} \sqrt{\frac{E_a^* + M^*}{E_a^*}} \; j_0\left(\frac{\omega_a}{R}r\right) \\ i\sqrt{\frac{E_a^* - M^*}{E_a^*}} \vec{\sigma}.\hat{r} \; j_1\left(\frac{\omega_a}{R}r\right) \end{array} \right) \Phi_{1/2}^m \zeta_\tau, \tag{2.39}$$

where

$$N^2 = \frac{A}{4\pi R^3 j_0^2(\omega_a)} \frac{E_a^*(E_a^* - M^*)R}{2E_a^{*2}R - 2E_a^* + M^*}. \tag{2.40}$$

By taking the explicit solutions of the Dirac equation

$$\psi_\alpha(\vec{r}) = N_\alpha \left(\begin{array}{c} \sqrt{\frac{E_a^* + M^*}{E_a^*}} \; j_{k\mp1}\left(\frac{\omega_a}{R}r\right) \\ i\sqrt{\frac{E_a^* - M^*}{E_a^*}} \vec{\sigma}.\hat{r} \; j_k\left(\frac{\omega_a}{R}r\right) \end{array} \right) \Phi_{km} \zeta_\tau, \tag{2.41}$$

where the upper (or lower) sign refers to k positive (or negative), it is easily verified that only $k = 1$ (or $k = -1$) leads to an angle-independent result on the right-hand side of Eq. (2.13). Thus only states with $j = 1/2$ can satisfy the nonlinear boundary condition as given.

2.3   Radius and Binding Energy of Nuclei

We have seen that the only change in the calculation of the energy in the MIT bag model for nuclei is the addition of a surface term, BS. It is assumed that B is a universal constant, chosen to fit one piece of data. Once B is chosen, because of the nonlinear boundary condition the radius of the bag is uniquely determined for each nuclei.

The meaning of this addition to energy-momentum tensor can be clarified by considering the total energy of the bag state,

$$P^0 \equiv E(A) = \int d^3x \; T_{\text{Bag}}^{00} = \int d^3x \; (T^{00}\theta_V + B\theta_S), \tag{2.42}$$

which we shall label $E(A)$ as a precursor to our discussion of binding energy later. Based on (2.35) and (2.42) the nuclear energy $E(A)$ is derived immediately

$$\begin{aligned} E(A) &= AE_a + 4\pi R^2 B \\ &= A\sqrt{\frac{\omega_a^2}{R^2} + M^{*2}} + AV_0 + 4\pi R^2 B. \end{aligned} \tag{2.43}$$

The first term is the kinetic energy, while the second is a surface term. Essentially it implies that it cost an energy BS to make this tension at the bag surface within which

---

[3]We use $\int_0^R dr \; r^2 j_m^2(\omega r/R) = \frac{R^3}{2}\left[j_m^2(\omega) + j_{m\pm1}^2(\omega) - \frac{2m+1}{\omega}j_m(\omega)j_{m\pm1}(\omega)\right].$

the baryons move. It should be intuitively clear that energy-momentum conservation is related to pressure balance at the bag surface, so that a small change in radius should not significantly increase $E(A)$. Nevertheless, the nonlinear boundary condition implies that

$$\frac{\partial E}{\partial R} = 0. \tag{2.44}$$

We wish to stress that it is an assumption of the model that B should be constant for all nuclei. As all nuclear bags have radii in the region $(1.0 \div 1.2)A^{1/3}$ fm, this assumption will be severely tested.

Generalizing Eq. (2.43) to include exited states, the nonlinear boundary condition implies

$$\frac{\partial E(A)}{\partial R} = -\frac{A\omega_a^2}{R^2\sqrt{\omega_a^2 + M^{*2}R^2}} + 8\pi RB = 0, \tag{2.45}$$

and hence

$$A = \frac{8\pi B}{\omega_a^2}R^3\sqrt{\omega_a^2 + M^{*2}R^2} \tag{2.46}$$

The real and positive root R of Eq. (2.46) is found out after an algebraic manipulation,

$$R = r_0 A^{1/3}, \tag{2.47}$$

where

$$
\begin{aligned}
r_0 &= \left(\frac{\omega_a}{4\pi B}\right)^{1/3}\alpha^{1/2}; \qquad n = 0, \quad k = -1, \tag{2.48}\\
\alpha &= \frac{(\beta/2)^{1/4}}{[1 - (\beta/2)^{3/2}]^{1/2} + (\beta/2)^{3/4}},\\
\beta &= \left[\left(\frac{256a}{27}\right)^{1/2} + 1\right]^{1/3} - \left[\left(\frac{256a}{27}\right)^{1/2} - 1\right]^{1/3},\\
a &= \left(\frac{AM^{*3}}{8\pi B\omega_a^2}\right)^2,
\end{aligned}
$$

(2.48) shows that r_0 actually depends weakly on A.

The above obtained formula (2.47) is well known in nuclear physics. It is one of the main successes of our model.

Using Eq. (2.47) we can then simplify the expression for $E(A)$:

$$E(A) = AV_0 + \left[\frac{\sqrt{\omega_a^2 + r_0^2 M^{*2}A^{2/3}}}{r_0} + 4\pi Br_0^2\right]A^{2/3}. \tag{2.49}$$

Hence the binding energy per nucleon is obtained

$$\varepsilon(A) = -(M - \phi_0) + \left[\frac{\sqrt{\omega_a^2 + r_0^2 M^{*2}A^{2/3}}}{r_0} + 4\pi Br_0^2\right]A^{-1/3}. \tag{2.50}$$

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Clearly, the remarkable result obtained for the binding energy per nucleon of the bag model of nuclei was indeed a coincidence.

As was known, the semi-empiric formula of Weizssacker [8, 11] for binding energy per nucleon reads

$$\varepsilon(A) = -a_1 + a_2 A^{-1/3} + a_3 \left(\frac{Z - N}{2A} \right)^2 + a_4 \frac{Z^2}{A^{4/3}}, \tag{2.51}$$

in which a_1, a_2, a_3, and a_4 take the following values, in the energy unit equal to 0.9311 MeV,

$$a_1 = 16.9177, \qquad a_2 = 19.120, \qquad a_3 = 101.777, \qquad a_4 = 0.7627.$$

Now let us indicate that (2.51) is possibly derived from our model if the parameters ϕ_0, V_0 and B are fitted adequately. Next confronting (2.50) with (2.51) we conclude that the above mentioned parameters must fulfil equalities,

$$M - V_0 = a_1, \tag{2.52}$$

$$\left[\frac{\sqrt{\omega_a^2 + r_0^2 M^{*2} R^2}}{r_0} + 4\pi B r_0^2 \right] = a_2. \tag{2.53}$$

It is worth to notice that parameter V_0 is explicitly defined by (2.52). The equation (2.53) constrains two unknown parameters of the theory, ϕ_0 and V_0.

The final term in Eq. (2.50) represents the contribution to the surface energy from the positive-frequency states, where the mass has been shifted by the constant, condensed scalar field ϕ_0. ϕ_0 (or the effective mass $M^* = M + \phi_0$ depends explicitly on the scalar field) is a dynamical quantity that must be calculated self-consistently using the thermodynamic argument that an isolated system at fixed A and V (and zero temperature) will minimize its energy:

$$\frac{\partial E(A, V; \phi_0)}{\partial \phi_0} = 0. \tag{2.54}$$

However, ϕ_0 is related to ω_a by the relation (2.34). As a consequence, (2.54) is replaced by conditions:

$$\frac{\partial}{\partial \omega_a} [E(A) + \lambda \varphi] = 0, \tag{2.55}$$

$$\frac{\partial}{\partial \phi_0} [E(A) + \lambda \varphi] = 0, \tag{2.56}$$

in which

$$\varphi = 1 - M^* R - \sqrt{\omega_a^2 + M^{*2} R^2} - \frac{\omega_a}{\tan \omega_a}.$$

Substituting λ from (2.55) and (2.56) one gets finally

$$M^* R = \frac{[M^* R + (\omega_a^2 + M^{*2} R^2)] \omega_a \tan^2 \omega_a}{\omega_a \tan^2 \omega_a + (\tan \omega_a - \omega_a - \omega_a \tan^2 \omega_a) \sqrt{\omega_a^2 + M^{*2} R^2}}, \tag{2.57}$$

with ϕ_0 and ω_a are the roots of the system (2.34) and (2.57).

This model problem is exactly solvable. It retains the essential features of the bag model for nuclei: A-dependence of nuclear radius and the formula for nuclear binding energy. Furthermore, it yields a simple solution to the field equation. This solution and model problem thus provide a meaningful starting point for describing the nuclear many-body system as well as a consistent basis for considering nuclei with $N \neq Z$ using relativistic nuclear physics, the bag model and standard many-body techniques.

We proceed to investigate nuclei with $N \neq Z$.

Chapter 3

Bag Model of Nuclei ($N \neq Z$)

To realistically discuss nuclei $N \neq Z$, it is necessary to extend bag model of nuclei $N = Z$ to include neutral field, which couple to the isovector current, and the coulomb interaction. If $Z \neq N$, the neutral charged, isovector field corresponding to $b_\mu^{(3)}$ can develop a classical, constant ground-state expectation value in nuclear matter according to

$$\langle b_\mu^{(3)} \rangle = \delta_{\mu 0} \delta^{j3} b_3.$$

The only change in the results of Chapter 2 is that there are now separate solutions for protons and neutrons, with the appropriate frequency modifications:

$$V_0 \rightarrow \begin{cases} V_0 + \frac{1}{2} b_3 + e A_0 & \text{for proton}, \\ V_0 + \frac{1}{2} b_3 & \text{for neutron}. \end{cases}$$

The field equation for uniform nuclear matter must also be extend to include contribution from the classical field b_3 and the coulomb potential A_0:

$$\left[i \gamma^\mu \partial_\mu - \gamma^0 V_0 - \frac{1}{2} \tau_3 \gamma^0 b_3 - \frac{1}{2} e (1 + \tau_3) \gamma^0 A_0 - (M + \phi_0) \right] \psi = 0.$$

The Lagrangian density, which is obtained from Eq. (2.8) by replacing the classical fields. Thus in bag model of nuclei with $N \neq Z$

$$\mathcal{L} = \overline{\psi} \left[i \gamma^\mu \partial_\mu - \gamma^0 V_0 - \frac{1}{2} \tau_3 \gamma^0 b_3 - \frac{1}{2} e (1 + \tau_3) \gamma^0 A_0 - (M + \phi_0) \right] \psi \theta_V + B \theta_S,$$

which is a generalization of Eq. (2.8) to allow for classical, constant fields ϕ_0, V_0, b_3, and A_0. Hence, the energy-momentum tensor is still (2.9).

3.1   Basic Assumptions

As was know, in Celenza and Shakin theory [2] the motion of nucleon in nuclear matter is described by the Dirac equation,

$$\left(i \gamma^\mu \partial_\mu - M - \Sigma \right) \psi(x_\mu) = 0, \tag{3.1}$$

where Σ is the nucleon self-energy having the form

$$\Sigma = \phi_0 + \gamma^0 V_0, \tag{3.2}$$

with ϕ_0 and V_0 constants.

Taking into account the isotopic degree of freedom and the Coulomb interaction of protons, in our model, we assume Σ has the generalized from

$$\Sigma = \phi_0 + \gamma^0 V_0 + \frac{1}{2}\tau_3\gamma^0 b_3 + \frac{1}{2}e(1+\tau_3)\gamma^0 A_0, \tag{3.3}$$

in which ϕ_0, V_0, b_3, and A_0 are constants. Inserting (3.3) into (3.1) we obtain the equation,

$$\left[i\gamma^\mu\partial_\mu - \gamma^0 V_0 - \frac{1}{2}\tau_3\gamma^0 b_3 - \frac{1}{2}e(1+\tau_3)\gamma^0 A_0 - (M + \phi_0)\right]\psi = 0, \tag{3.4}$$

where the mass matrix M is, of course,

$$M = \begin{pmatrix} M_p & 0 \\ 0 & M_n \end{pmatrix}$$

Now our basic assumption is formulated: the nucleus A is considered to be a M.I.T. bag, inside which the motion of nucleon is described by the Dirac equation (3.4). The parameters ϕ_0, V_0, b_3, and A_0 will be fitted to experimental data.

Let us next consider the energy-momentum conservation for nuclear bag. Let $T^{\mu\nu}$ be the energy-momentum tensor of nucleon, described by Eq. (3.4), inside the bag. Then the total energy-momentum tensor $T^{\mu\nu}_{\text{Bag}}$ of the nucleus is clearly given by

$$T^{\mu\nu}_{\text{Bag}} = T^{\mu\nu}\theta_V + B\theta_S, \tag{3.5}$$

where θ_V and θ_S are the well-known step functions for volume and surface of the bag, respectively, and B is the surface tension.

From (3.5) it follows that

$$\partial_\mu T^{\mu\nu}_{\text{Bag}} = \left[\frac{1}{2}\, n.\partial(\overline{\psi}\psi)\big|_S + 2B\right]n^\nu\Delta_S.$$

The energy-momentum conservation,

$$\partial_\mu T^{\mu\nu}_{\text{Bag}} = 0,$$

leads to

$$B = -\frac{1}{4}\, n.\partial(\overline{\psi}\psi)\big|_S, \tag{3.6}$$

which resembles the nonlinear boundary condition in the bag model for baryon [3].

Equation (3.4) and the relation (3.6) constitute the basic ingredients of out model.

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3.2 *A*-dependence of Nuclear Radius

The solution of (3.4) can be found in the form

$$\psi(x_\mu) = \begin{pmatrix} \psi_p(x_\mu) \\ \psi_n(x_\mu) \end{pmatrix}, \tag{3.7}$$

$$\begin{aligned}
\psi_p(x_\mu) &= \psi_p(\vec{x})e^{-iE_p t}, \\
\psi_n(x_\mu) &= \psi_n(\vec{x})e^{-iE_n t}.
\end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.4) we arrive at the equations for protons and neutron, separately,

$$\left[-i\vec{\alpha}.\nabla + V_0 + \frac{1}{2}b_3 + eA_0 + \beta(M_p + \phi_0) \right] \psi_p(\vec{x}) = E_p\psi_p(\vec{x}), \tag{3.9}$$

$$\left[-i\vec{\alpha}.\nabla + V_0 - \frac{1}{2}b_3 + \beta(M_n + \phi_0) \right] \psi_n(\vec{x}) = E_n\psi_n(\vec{x}). \tag{3.10}$$

For convenience, let us define

$$\begin{aligned}
M_p^* &= M_p + \phi_0, \\
M_n^* &= M_n + \phi_0, \\
E_p^* &= E_p - V_0 - \frac{1}{2}b_3 - eA_0, \\
E_n^* &= E_n - V_0 + \frac{1}{2}b_3.
\end{aligned}$$

With this in mind we rewrite (3.9) and (3.10) as follows

$$\begin{aligned}
\left[-i\vec{\alpha}.\nabla + \beta M_p^* \right] \psi_p(\vec{x}) &= E_p^*\psi_p(\vec{x}), \tag{3.11} \\
\left[-i\vec{\alpha}.\nabla + \beta M_n^* \right] \psi_n(\vec{x}) &= E_n^*\psi_n(\vec{x}). \tag{3.12}
\end{aligned}$$

It is known that the solutions of (3.11) and (3.12) read, respectively,

$$\psi_p(\vec{x}) = N_p \begin{pmatrix} \sqrt{\frac{E_p^* + M_p^*}{E_p^*}}\, j_{k-1}(W_p r)\Phi_{km} \\ i\sqrt{\frac{E_p^* - M_p^*}{E_p^*}}\, j_k(W_p r)\Phi_{-km} \end{pmatrix}, \tag{3.13}$$

$$\psi_n(\vec{x}) = N_n \begin{pmatrix} \sqrt{\frac{E_n^* + M_n^*}{E_n^*}}\, j_{k-1}(W_n r)\Phi_{km} \\ i\sqrt{\frac{E_n^* - M_n^*}{E_n^*}}\, j_k(W_n r)\Phi_{-km} \end{pmatrix}, \tag{3.14}$$

where k and $k-1$ are the indices of the eigenfunctions corresponding to the eigenvalues of operator K,

$$\begin{aligned}
K &= \beta(\vec{\sigma}.\vec{l} + 1), \\
W_p^2 &= E_p^{*2} - M_p^{*2}, \\
W_n^2 &= E_n^{*2} - M_n^{*2},
\end{aligned} \tag{3.15}$$

and the normalization constants N_p, N_n are defined by

$$\int d^3x \; \psi_p^\dagger \psi_p \theta_V \;\; = \;\; Z, \tag{3.16}$$

$$\int d^3x \; \psi_n^\dagger \psi_n \theta_V \;\; = \;\; N, \tag{3.17}$$

Z and N are the numbers of protons and neutrons contained in the nucleus A.

Now let us assume that the bag has a spherical shape with radius R. Then the boundary condition of the M.I.T. bag model is used, which provides the eigenfrequencies of proton and neutron, Ω_p and Ω_n, correspondingly,

$$\begin{aligned}
\Omega_p &= W_p R, \\
\Omega_n &= W_n R.
\end{aligned} \tag{3.18}$$

As was known, Ω_p and Ω_n satisfies the equations

$$\tan \Omega_p \;\; = \;\; \frac{\Omega_p}{1 - M_p^* R - \sqrt{\Omega_p^2 + M_p^{*2} R^2}}, \tag{3.19}$$

$$\tan \Omega_n \;\; = \;\; \frac{\Omega_n}{1 - M_n^* R - \sqrt{\Omega_n^2 + M_n^{*2} R^2}}. \tag{3.20}$$

It is worth to remember that (3.19) and (3.20) are derived for $k = 1$, to which correspond the only states satisfying (3.6). Taking into consideration (3.15) and (3.18) we get the energy spectra for proton and neutron, respectively,

$$E_p \;\; = \;\; \pm\sqrt{\frac{\Omega_p^2}{R^2} + M_p^{*2}} + V_0 + \frac{1}{2}b_3 + eA_0, \tag{3.21}$$

$$E_n \;\; = \;\; \pm\sqrt{\frac{\Omega_n^2}{R^2} + M_n^{*2}} + V_0 - \frac{1}{2}b_3. \tag{3.22}$$

For convenience, the sign $(-)$ drops out in what follows.

Based on (3.21), (3.22), and (3.5) the nuclear energy $E(A)$ is derived immediately

$$E(A) = Z\sqrt{\frac{\Omega_p^2}{R^2} + M_p^{*2}} + N\sqrt{\frac{\Omega_n^2}{R^2} + M_n^{*2}} + AV_0 + \frac{Z-N}{2}b_3 + eZA_0 + 4\pi BR^2. \tag{3.23}$$

Next let us introduce the mean mass $\langle M \rangle$, the effective mass M^* and the mean frequency $\langle \Omega \rangle$ of nucleons contained in nucleus A,

$$\begin{aligned}
\langle M \rangle &= \frac{ZM_p + NM_n}{A}, \\
M^* &= \langle M \rangle + \phi_0, \\
A\sqrt{\frac{\langle \Omega \rangle^2}{R^2} + M^{*2}} &= Z\sqrt{\frac{\Omega_p^2}{R^2} + M_p^{*2}} + N\sqrt{\frac{\Omega_n^2}{R^2} + M_n^{*2}}.
\end{aligned} \tag{3.24}$$

It is easily prove that $\langle\Omega\rangle$, defined by (3.24), really exists.

Substituting of (3.24) into (3.23) leads to

$$E(A) = A\sqrt{\frac{\langle\Omega\rangle^2}{R^2} + M^{*2}} + AV_0 + \frac{Z - N}{2}b_3 + eZA_0 + 4\pi BR^2. \qquad (3.25)$$

The nonlinear boundary condition (3.6) requires

$$\frac{\partial E(A)}{\partial R} = 0,$$

which yields

$$A = \frac{8\pi B}{\langle\Omega\rangle}R^3\sqrt{1 + \left(\frac{M^*R}{\langle\Omega\rangle}\right)^2}. \qquad (3.26)$$

The real and positive root R of (3.26) is found out after an algebraic manipulation,

$$R = r_0 A^{1/3}, \qquad (3.27)$$

where

$$r_0 = \left(\frac{\langle\Omega\rangle}{4\pi B}\right)^{1/3}\delta^{1/2}, \qquad (3.28)$$

$$\delta^{1/2} = \frac{(\xi/2)^{1/4}}{[1 - (\xi/2)^{3/2}]^{1/2} + (\xi/2)^{3/4}},$$

$$\xi = \left[\left(\frac{256a}{27} + 1\right)^{1/2} + 1\right]^{1/3} - \left[\left(\frac{256a}{27} + 1\right)^{1/2} - 1\right]^{1/3},$$

$$a = \left(\frac{AM^{*3}}{8\pi B\langle\Omega\rangle}\right)^2,$$

(3.28) shows that r_0 actually depends weakly on A.

The above obtained formula (3.27) is well known in nuclear physics. It is one of the main successes of our model.

Finally, the normalization constant N_p and N_n given by (3.13) and (3.14), are calculated for $k = 1$,

$$N_p = \left(\frac{Z}{4\pi R^3}\right)^{1/2}\frac{\left[E_p^*(E_p^* - M_p^*)R\right]^{1/2}}{j_0(\Omega_p)\left[2E_p^{*2}R - 2E_p^* + M_p^*\right]^{1/2}},$$

$$N_n = \left(\frac{N}{4\pi R^3}\right)^{1/2}\frac{\left[E_n^*(E_n^* - M_n^*)R\right]^{1/2}}{j_0(\Omega_n)\left[2E_n^{*2}R - 2E_n^* + M_n^*\right]^{1/2}} \qquad (3.29)$$

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3.3   Weizssacker Formula

As was known, the semi-empiric formula of Weizssacker [7, 10] for binding energy per nucleon reads

$$f = -a_1 + a_2 A^{-1/3} + a_3 \frac{(Z-N)^2}{4A^2} + a_4 \frac{Z^2}{A^{4/3}}, \tag{3.30}$$

in which a_1, a_2, a_3, and a_4 take the following values, in the energy unit equal to 0.9311 MeV,

$$a_1 = 16.9177, \qquad a_2 = 19.120, \qquad a_3 = 101.777,$$

and

$$a_4 = \frac{3e^2}{5r_0} = 0.7627.$$

The charge distribution radius r_0 for almost nuclei is fitted to be

$$r_c = 1.2162 \ 10^{-13} \ \text{cm},$$

(3.30) agrees well with experimental data for most nuclei.

Now let us indicate that (3.30) is possibly derived from our model if the parameters ϕ_0, V_0, b_3, A_0, and B are fitted adequately. For this end, let us substitute (3.27) into (3.25),

$$E(A) = AV_0 + \left[\frac{\sqrt{\langle\Omega\rangle^2 + r_0^2 M^{*2} A^{2/3}}}{r_0} + 4\pi B r_0^2 \right] A^{2/3} + \frac{b_3(Z-N)}{2} + A_0 eZ. \tag{3.31}$$

Therefrom, the binding energy per nucleon is obtained

$$f = V_0 - \langle M \rangle + \left[\frac{\sqrt{\langle\Omega\rangle^2 + r_0^2 M^{*2} A^{2/3}}}{r_0} + 4\pi B r_0^2 \right] A^{-1/3} + \frac{b_3(Z-N)}{2A} + A_0 \frac{eZ}{A}. \tag{3.32}$$

Next confronting (3.32) with (3.30) we conclude that the above mentioned parameters must fulfil equalities

$$V_0 - \langle M \rangle \;=\; -a_1, \tag{3.33}$$

$$\frac{\sqrt{\langle\Omega\rangle^2 + r_0^2 M^{*2} A^{2/3}}}{r_0} + 4\pi B r_0^2 \;=\; a_2, \tag{3.34}$$

$$b_3 \;=\; a_3 \frac{Z-N}{2A}, \tag{3.35}$$

$$A_0 \;=\; a_4 \frac{Z}{e A^{1/3}} = \frac{3r_0}{5r_c} \frac{eZ}{R}. \tag{3.36}$$

It is clear that (3.33) and (3.36) express directly the physical meaning of b_3 and A_0:

– The mean field value b_3 is proportional to the relative ratio of the numbers of protons and neutrons, contained in nucleus A.

– The mean value of Coulomb potential created by Z protons equals to that created by a sphere of charge Ze, embedded in a nuclear medium, the dielectric coefficient of which is $3r_0/5r_c$.

It is worth to notice that three parameters V_0, b_3, and A_0 are explicitly defined by (3.33), (3.35), and (3.36). The equation (3.34) contains two unknown parameters of the theory, ϕ_0 and B. As was shown in the Walecka theory [1], ϕ_0 is a dynamical quality and therefore it is defined self-consistently. Namely, we use the thermodynamic argument that an isolated system with fixed baryon number A and volume V will minimize its energy,

$$\frac{\partial E(A, V; \phi_0)}{\partial \phi_0} = 0. \tag{3.37}$$

However, ϕ_0 is related to Ω_p and Ω_n by the relations (3.19) and (3.20). As a consequence, (3.37) is replaced by the conditions:

$$\frac{\partial}{\partial \Omega_p} \left[E(A) + \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \right] = 0, \tag{3.38}$$

$$\frac{\partial}{\partial \Omega_n} \left[E(A) + \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \right] = 0, \tag{3.39}$$

$$\frac{\partial}{\partial \phi_0} \left[E(A) + \alpha_1 \varphi_1 + \alpha_2 \varphi_2 \right] = 0. \tag{3.40}$$

in which

$$\varphi_1 = 1 - M_p^* R - \sqrt{\Omega_p^2 + M_p^{*2} R^2} - \frac{\Omega_p}{\tan \Omega_p},$$

$$\varphi_1 = 1 - M_n^* R - \sqrt{\Omega_n^2 + M_n^{*2} R^2} - \frac{\Omega_n}{\tan \Omega_n}.$$

Eliminating α_1 and α_2 from (3.38) and (3.39) and substituting them into (3.40) one gets finally

$$\frac{2Z\Omega_p \sin^2 \Omega_p}{\sin^2 \Omega_p + 2\Omega_p} - \frac{ZM_p^* R\sqrt{\Omega_p^2 + M_p^{*2} R^2}}{M_p^* R + \sqrt{\Omega_p^2 + M_p^{*2} R^2}} + \frac{2N\Omega_n \sin^2 \Omega_n}{\sin^2 \Omega_n + 2\Omega_n} - \frac{NM_n^* R\sqrt{\Omega_n^2 + M_n^{*2} R^2}}{M_n^* R + \sqrt{\Omega_n^2 + M_n^{*2} R^2}} = 0, \tag{3.41}$$

ϕ_0 and Ω_p, Ω_n are the roots of the system (3.19), (3.20) and (3.41).

Chapter 4

Conclusion and Discussion

In the previous sections the basic assumptions and the general results of our bag model for nuclei are presented in detail. The model is built on a simple hypothesis: the nucleus is considered to be a MIT bag, in which the motion of nucleons is described by the Dirac equation and the mean field values ϕ_0, V_0, b_3, and A_0 are supposed to be constants. In addition to these mean fields, there exists the surface tension B of the bag that guarantees the energy-momentum conservation.

Two major successes are: the formula (3.27) for the nuclear radius R and the Weizsacker formula with the parameters verified in (3.33-36). All the parameters appearing in the theory ϕ_0, V_0, b_3, A_0, and B are, in principle, determined by the equations (3.33-36), (3.19, 20), and (3.41). Thus, our theory is a mathematically closed system. The development of the formalism suggested here will be carried out for various concrete nuclei in next papers.

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Bibliography

[1] B. D. Serot and J. D. Walecka, *Adv. in Nucl. Phys.*, **16**, 1, (1986).

[2] L. S. Calenza and C. M. Shakin, *Relativistic Nuclear Physics*, World Scientific, (1986).

[3] A. W. Thomas, *Adv. in Nucl. Phys.*, **13**, 1, (1983).

[4] A. Arima and F. Iachello, *Ann. Phys.* (N.Y.), **99**, 253, (1976).

[5] Y. K. Gambhir, P. Ring, and A. Thicnet, *Ann. Phys.*, **198**, 132, (1990).

[6] I. Tanihata, T. Kobayashi, S. Shimaura, and T. Minaminoso, *Proceedings of I Intern. Conf. on PRadioactive Nuclear Beams*, Berkerley, (1989), p. 429.

[7] D. Hirata, H. Toki, T. Watabe, I. Tanihata, and B. V. Calson, *Phys. Rev.* **C44**, 1467, (1991).

[8] C. F. Weizssacker, *Zs. f. Phys.*, **96**, 431, (1935).

[9] P. A. Seeger and W. M. Haward, *Nucl. Phys.*, **A238**, 491, (1975).

[10] A. E. S. Green, *Phys. Rev.*, **95**, 1006, (1954).

[11] Neumark, *Solution of Cubic and Quartic Equation*, (1965).